

Nonabelian solutions in a Melvin magnetic universe

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February 2, 2008

Abstract

We show the existence of $D = 4$ nonabelian solutions approaching asymptotically a dilatonic Melvin spacetime background. An exact solution generalizing the Chamseddine-Volkov soliton for a nonzero external $U(1)$ magnetic field is also reported.

1 Introduction

According to the so called "no-hair" conjecture, an asymptotically flat, stationary black hole is uniquely described in terms of a small set of asymptotically measurable quantities. However, in recent years counterexamples to this conjecture were found in several theories, most of them containing nonabelian matter fields. The first nonabelian "hairy" black hole solutions within the framework of $SU(2)$ Einstein-Yang-Mills (EYM) theory, were presented in [1]. Although the new solutions were static and have vanishing Yang-Mills (YM) charges, they were different from the Schwarzschild black hole and, therefore, not characterized by their total mass. Remarkably, in the limit of zero event horizon radius of these configurations, the globally regular, particle-like solutions originally found in [2] are recovered.

It is however worth inquiring what happens with these solutions if we drop the assumption of asymptotic flatness. The asymptotically (anti)-de Sitter solutions which are found for a nonzero cosmological constant enjoyed considerable attention in the last years and present many interesting features [3].

Another interesting example of non-asymptotically flat solutions in general relativity is given by the Melvin magnetic universe, describing a bundle of magnetic flux lines in gravitational-magnetostatic equilibrium [4]. This solution has a number of interesting features, providing the closest approximation in general relativity for an uniform magnetic field. The nonsingular nature of this solution (at the cost of losing the asymptotic flatness) motivated Melvin to refer to his solution as a magnetic "geon".

There exists a fairly extensive literature on the properties of this magnetic universe, starting with a study by Thorne, which investigates also its dynamical behaviour under arbitrary large radial perturbations [5]. Various generalizations of this type of solution have been proposed (see [6] for a review and relevant references), particularly interesting being the Melvin solution in Kaluza-Klein (KK) theory. This configuration derives from a flat five dimensional spacetime by performing a $U(1)$ reduction with a twist in the identifications [7, 8], the four dimensional theory containing an extra dilaton field. An exact solution of Einstein-Maxwell equations describing a black hole in a background Melvin universe was constructed by Ernst [9], and admits also a straightforward generalization in the KK case [7].

It is therefore natural to ask whether the well-known hairy black hole solutions admit generalizations with Melvin-type asymptotics and what new effects emerge due to the presence of a background magnetic field. The main purpose of this paper is to present such solutions in EYM-Higgs- $U(1)$ -dilaton theory, which approach asymptotically a Melvin background.

2 General framework

We consider the following action in four spacetime dimensions

$$I_4 = \frac{1}{4\pi} \int d^4x \sqrt{-\gamma} \left[\frac{\mathcal{R}}{4} - \frac{1}{2} \nabla_i \psi \nabla^i \psi - e^{2a\psi} \frac{1}{4} f_{ij} f^{ij} - e^{2a\psi/3} \frac{1}{4} \mathcal{F}'^I_{ij} \mathcal{F}'^{Iij} - e^{-4a\psi/3} \frac{1}{4} D_i \Phi^I D^i \Phi^I \right], \quad (1)$$

which describes a gravitating system with a scalar triplet Φ^I ($I = 1, 2, 3$), an SU(2) Yang-Mills (YM) potential \mathcal{A}^I_i (with field strength $\mathcal{F}^I_{ij} = \partial_i \mathcal{A}^I_j - \partial_j \mathcal{A}^I_i + \epsilon^{IJK} \mathcal{A}^J_i \mathcal{A}^K_j$), an abelian potential \mathcal{W}_i ($f_{ij} = \partial_i \mathcal{W}_j - \partial_j \mathcal{W}_i$ being the corresponding field strength) and a dilaton field ψ , a being the dilaton coupling constant, and we note $\mathcal{F}'^I_{ij} = \mathcal{F}^I_{ij} + 2\Phi^I f_{ij}$. This expression of the action has a higher dimensional origin and is motivated in the next Section.

2.1 The dilaton Melvin solution

The Melvin solution in Einstein-Maxwell-dilaton theory is found for vanishing SU(2) and triplet scalar fields, $\mathcal{F}_{ij} = \Phi^I = 0$, and reads [7]

$$ds^2 = \Lambda^{\frac{2}{1+a^2}} (dr^2 + r^2 d\theta^2 - dt^2) + \Lambda^{-\frac{2}{1+a^2}} r^2 \sin^2 \theta d\varphi^2, \quad \text{with } \Lambda = 1 + \left(\frac{1+a^2}{4}\right) B_0^2 r^2 \sin^2 \theta, \quad (2)$$

the dilaton and the U(1) potential being

$$e^{2a(\psi-\psi_0)} = \Lambda^{\frac{2a^2}{1+a^2}}, \quad \mathcal{W}_i dx^i = \frac{e^{-a\psi_0} B_0 r^2 \sin^2 \theta}{2\Lambda} d\varphi. \quad (3)$$

The solution is parametrized by ψ_0 , the value of the scalar field on the symmetry axis and B_0 , which characterizes the central strength of the magnetic field. Although not asymptotically flat, the geometry of this solution is singularity free and geodesically complete. A curious property of (2)-(3) is that the total flux

$$\Phi_m = \oint_{\infty} \mathcal{W}_\varphi = e^{-a\psi_0} \frac{4\pi}{1+a^2} \frac{1}{B_0} \quad (4)$$

is finite and inversely proportional to B_0 . (The total magnetic flux for this cylindrically symmetric solution is obtained by integrating over the entire physical area perpendicular to the z -axis, with $z = r \cos \theta$ [4]). However, in the limit $B_0 \rightarrow 0$, even if the geometry becomes flat and the field strength goes to zero at the centre, the total flux diverges.

The solution describing a Schwarzschild black hole immersed in the dilatonic Melvin universe is a straightforward generalization of (2), (3) and has a line element [7]

$$ds^2 = \Lambda^{\frac{2}{1+a^2}} \left(\frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 - \left(1 - \frac{2M}{r}\right) dt^2 \right) + \Lambda^{-\frac{2}{1+a^2}} r^2 \sin^2 \theta d\varphi^2, \quad (5)$$

with the same expressions for Λ , dilaton and U(1) potential (the $\psi = 0$ case was first discussed in [9]).

The constant M which enters the line element (5) corresponds to the black hole's mass. This axially symmetric solution contains an event horizon at $r = 2M$ as in the Schwarzschild vacuum case, but is not asymptotically flat owing to the gravitational effects of the magnetic field. It is evident that standard Kruskal coordinates may be introduced in order to extend the solution across the event horizon, the only singularity occurring at $r = 0$. More details on this solution can be found e.g. in [6],[10],[11].

2.2 Nonabelian Einstein-Yang-Mills-dilaton solutions

For a vanishing U(1) and triplet scalar fields, $\mathcal{W}_i = \Phi^I = 0$, one finds a different class of solutions, corresponding to dilaton generalizations of the SU(2)-EYM hairy black holes [1]. In the simplest, spherically symmetric case, these configurations are usually described by using a line element

$$ds^2 = \frac{dr^2}{N(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - \sigma^2(r) N(r) dt^2, \quad \text{with } N(r) = 1 - \frac{2m(r)}{r}, \quad (6)$$

$m(r)$ corresponding to the total mass-energy within the radius r , and a $SU(2)$ nonabelian potential

$$\mathcal{A}_i dx^i = w(r)(\tau_1 d\theta + \tau_2 \sin\theta d\varphi) + \tau_3 \cos\theta d\varphi, \quad (7)$$

τ_i being the Pauli matrices. The metric functions $m(r)$, $\sigma(r)$, the gauge potential function $w(r)$ and the dilaton function $\psi(r)$ are solutions of the equations

$$\begin{aligned} m' &= 2\left(e^{2a\psi/3}\left(w'^2 N + \frac{(w^2 - 1)^2}{2r^2}\right) + \frac{r^2}{2}N\psi'^2\right), & \sigma' &= \frac{2}{r}\left(e^{2a\psi/3}w'^2 + \frac{1}{2}\psi'^2 r^2\right), \\ \left(\sigma e^{2a\psi/3}Nw'\right)' &= \sigma e^{2a\psi/3}\frac{w(w^2 - 1)}{r^2}, & (Nr^2\sigma\psi')' &= \frac{2a}{3}\sigma e^{2a\psi/3}\left(w'^2 N + \frac{(w^2 - 1)^2}{2r^2}\right), \end{aligned} \quad (8)$$

(where a prime denotes a derivative with respect to the radial coordinate r), with suitable boundary conditions.

Although no exact nonabelian solutions of the above equations are known, Refs. [12, 13] present both analytical and numerical arguments for the existence of a discrete family of black hole solutions uniquely characterized by the number of nodes p of the function $w(r)$, with $p \geq 1$. Nontrivial solutions are found for any value of the dilaton coupling constant a , the dilaton field vanishing asymptotically.

These solutions approach asymptotically the Minkowski spacetime background ($m(r) \rightarrow M$, $\sigma(r) \rightarrow 1$) and have no global nonabelian charge (although their dilaton charge is nonzero). The black hole configurations exist for any value of the event horizon radius r_h . The gauge potential w interpolates between $w(r_h) = w_0$ (with $|w_0| < 1$) and $w(r \rightarrow \infty) = \pm 1$, the Schwarzschild solution being recovered for $w(r) = \pm 1$ (a pure gauge field), $\sigma = 1$, $\psi = 0$ and $m(r) = M$. In the limit $r_h \rightarrow 0$, a dilatonic generalization of the Bartnik-McKinnon EYM solutions [2] is approached.

The thermodynamics of the EYM-dilaton black holes can be discussed in the standard way (see e.g. [14]); it turns out that their entropy is one quarter of the event horizon area $S = \pi r_h^2$, while their Hawking temperature is $T_H = \sigma(r_h)N'(r_h)/(4\pi)$.

3 The twisting procedure and new solutions

The purpose of this Section is to present a family of solutions which extremizes the action (1), keeping the basic features of both the Melvin universe (2) and the nonabelian solutions (6), (7).

Here we restrict to the case of a dilaton coupling constant $a = \sqrt{3}$, in which case the nonabelian solutions (6) can be uplifted to become solutions of the $SU(2)$ EYM equation in five dimensions [15, 16], extremizing the action

$$I_5 = \frac{1}{4\pi} \int d^5x \sqrt{-g} \left(\frac{1}{4}R - \frac{1}{4}F_{\mu\nu}^I F^{I\mu\nu} \right). \quad (9)$$

In a five-dimensional perspective, the solutions of the $D = 4$ EYMd equations (8) with $a = \sqrt{3}$ describe hairy black strings or nonabelian vortices, with a line element (x^5 being the extra-direction which is supposed to be compact and with a unit length)

$$ds_5^2 = e^{-a\psi} \left(\frac{dr^2}{N} + r^2(d\theta^2 + \sin^2\theta d\varphi) - \sigma^2 N dt^2 \right) + e^{2a\psi} (dx^5)^2, \quad (10)$$

and the same $SU(2)$ ansatz (7), i.e. a vanishing fifth component of the nonabelian potential, $A_5 = 0$.

The way to introduce a $D = 4$ magnetic field in a KK setup involves twisting the compactification direction. Following [7, 8] one shifts the coordinate $\varphi \rightarrow \varphi + B_0 x^5$ (with B_0 an arbitrary real constant), and reidentifies points appropriately. The next step is to consider the KK reduction with respect to the Killing vector $\partial/\partial x^5$, according to the generic prescription

$$ds_5^2 = e^{-a\psi} \gamma_{ij} dx^i dx^j + e^{2a\psi} (dx^5 + 2\mathcal{W}_i dx^i)^2, \quad (11)$$

$\gamma_{ij}dx^i dx^j$ being the four dimensional line element and \mathcal{W}_i the U(1) potential. For the reduction of the YM action term, a convenient $D = 5$ SU(2) ansatz is

$$A_\mu^I dx^\mu = \mathcal{A}_i^I dx^i + \Phi^I (dx^5 + 2\mathcal{W}_i dx^i), \quad (12)$$

where \mathcal{A}_i^I is a purely four-dimensional YM gauge field potential, while Φ^I corresponds after the dimensional reduction to a triplet Higgs field. It can be verified that the KK reduction of the action (9) with respect to the x^5 -direction, taken according to (11), (12), yields the four-dimensional action (1).

Therefore, upon reduction, the new $D = 4$ solutions based on the configurations in Section 2.2, have a line element

$$ds^2 = \sqrt{\Lambda} \left(\frac{dr^2}{N} + r^2 d\theta^2 - \sigma^2 N dt^2 \right) + \frac{r^2 \sin^2 \theta}{\sqrt{\Lambda}} d\varphi^2, \quad \text{with } \Lambda = 1 + e^{-3a\psi} B_0^2 r^2 \sin^2 \theta, \quad (13)$$

the only nonvanishing component of the U(1) potential vector \mathcal{W}_i being

$$\mathcal{W}_\varphi = \frac{e^{-3a\psi} B_0 r^2 \sin^2 \theta}{2\Lambda}. \quad (14)$$

The new $D = 4$ dilaton field $\bar{\psi}$ is

$$\bar{\psi} = \psi + \frac{1}{2a} \log \Lambda, \quad (15)$$

while the four dimensional YM field is given by

$$\mathcal{A}_i dx^i = w(\tau_1 d\theta + \tau_2 \sin \theta d\varphi) + \tau_3 \cos \theta d\varphi - 2B_0(\tau_2 w \sin \theta + \tau_3 \cos \theta) \mathcal{W}_\varphi d\varphi. \quad (16)$$

Different from the seed solutions, the new configurations have a nonvanishing Higgs field

$$\Phi = B_0(w \sin \theta \tau_2 + \cos \theta \tau_3). \quad (17)$$

It can easily be seen that for a vanishing nonabelian matter content ($w = \pm 1$), the dilatonic Ernst solution [9] describing a Schwarzschild black hole in a Melvin background is recovered, while setting $B_0 = 0$ leads us back to the EYMd seed solution (6), (7).

A different type of configuration is found for $w(r) = 0$, describing a magnetic monopole black hole placed in a Melvin universe (note that here the four dimensional geometry has a closed form expression [7], for a different parametrization instead of (6), however).

For the generic case, one can see that the causal structure of the seed EYMd solution is not changed by the twisting procedure. Supposing one starts with an initial EYMd hairy black hole solution, one finds that the Melvin-type metric (13) describes, in terms of the usual definitions, a black hole, with an event horizon and trapped surfaces. It has a horizon located at $r = r_h$ (where $N(r_h) = 0$), which is independent of the value of the magnetic field strength. A globally regular configuration (which differs from the Melvin solution) is found in the limit of zero event horizon radius. For $r \rightarrow \infty$, the line element (13) approaches the Melvin background (2) with $a = \sqrt{3}$.

Similar to the initial YM ansatz (7), the YMH fields (16), (17) are written in a singular gauge. A regular form is obtained after applying a gauge transformation $S = e^{i\pi\tau_3/4} e^{i\theta\tau_2/2} e^{i\varphi\tau_3/2}$. Their new expression, written in terms of a general ansatz used before in the literature on axially symmetric nonabelian solutions (see e.g. [19, 20]) is

$$A_\mu dx^\mu = \left[\frac{H_1}{r} dr + (1 - H_2) d\theta \right] \tau_\varphi - \sin \theta [H_3 \tau_r + (1 - H_4) \tau_\theta] d\varphi, \quad \Phi = (\phi^r \tau_r + \phi^\theta \tau_\theta), \quad (18)$$

where $H_1 = 0$, $H_2 = w$, $H_3 = -2B_0 \mathcal{W}_\varphi \cot \theta$, $H_4 = w(1 - 2\mathcal{W}_\varphi B_0)$ and $\phi^r = B_0 \cos \theta$, $\phi^\theta = -wB_0 \sin \theta$. As usual, the symbols τ_r , τ_θ and τ_ϕ in the above relation denote the dot products of the cartesian vector

of Pauli matrices, $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$, with the spatial unit vectors $\vec{e}_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, $\vec{e}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$, $\vec{e}_\phi = (-\sin \phi, \cos \phi, 0)$, respectively.

The matter fields of the new solution possess a nontrivial dependence on the polar coordinate θ . The modulus of the Higgs field $|\Phi| = \sqrt{\Phi^I \Phi^I}$ approaches a constant value at infinity and vanishes on p circles in the xy -plane ($\theta = \pi/2$), which are located at the zeros of the seed gauge potential $w(r)$. (The occurrence of asymptotically flat vortex ring solutions in a pure EYM theory has been noticed in [17] for a set of monopole-antimonopole solutions). Given the non asymptotically flat character of the spacetime, the interpretation of the matter field configurations in this solution is not obvious. However, since the modulus of the Higgs field is constant at infinity, as in the asymptotically flat case, we suggest that the 't Hooft electromagnetic field strength tensor

$$F_{\mu\nu} = \varepsilon_{IJK} \hat{\Phi}^I \partial_\mu \hat{\Phi}^J \partial_\nu \hat{\Phi}^K + \partial_\mu (\hat{\Phi}^I A_\nu^I) - \partial_\nu (\hat{\Phi}^I A_\mu^I), \quad (19)$$

(where $\hat{\Phi}^I$ is the normalized Higgs field) might be used to analyze the solutions. Then, following [18], one would evaluate the total nonabelian magnetic charge of the configurations, by integrating the 't Hooft electromagnetic field strength tensor,

$$F_{\theta\varphi} = ((1 - 2B_0 \mathcal{W}_\varphi) \sqrt{w^2 \sin^2 \theta + \cos^2 \theta})_{,\theta}. \quad (20)$$

Thus the magnetic charge inside a closed surface \mathcal{S} would be expressed as $m = \frac{1}{V(\mathcal{S})} \int_{\mathcal{S}} F_{\mu\nu} dx^\mu dx^\nu$, which turns out to vanish for the new solution (although locally the magnetic charge density would be nonzero).

To interpret the new solution we now suggest to consider the asymptotic expansion of the function $w(r) = \pm (1 - \frac{c}{r} \dots)$ in the 't Hooft field strength tensor (20), yielding $F_{\theta\varphi} = ((1 - 2B_0 \mathcal{W}_\varphi)(1 - \frac{c \sin^2 \theta}{r} + O(\frac{1}{r^2})))_{,\theta}$ and compare with the gauge potential of a magnetic dipole with dipole moment μ , $\tilde{\mathcal{W}}_\varphi = \frac{\mu \sin^2 \theta}{r}$ [17, 18]. The analogous functional dependence then hints at the possibility to interpret the new solution as a magnetic dipole with dipole moment $\mu = -c$, immersed in a Melvin background. In the asymptotically flat case discussed in [17], the vortex ring solutions (where the Higgs field vanishes on one or more rings) analogously correspond to magnetic dipoles.

A computation of the thermodynamic properties of the solution (13)-(17) can be done by applying the same approach as for the $B_0 = 0$ case. The computation of the mass and total Euclidean action is done with respect to the Melvin background (2) (with $a = \sqrt{3}$). The instanton that enters the calculation of the gravitational action is obtained by setting $\tau = it$ in (13). Similar to the pure Einstein-Maxwell-dilaton case [11], it follows that the thermodynamic properties of these black holes are not affected by the background U(1) magnetic field. In particular we find the same entropy and mass as for the asymptotically flat configurations; the value of the Hawking temperature is also unchanged. A similar behaviour has been noticed in [11] for the Ernst solution (5). Therefore, this seems to be a generic property of static black hole solutions in a background U(1) magnetic field extending to infinity.

4 Chamseddine-Volkov soliton in a background magnetic field

The procedure above may be applied to other nonabelian solutions with a higher dimensional origin. A particularly interesting case is given by the Chamseddine-Volkov solution [21, 22] of the $\mathcal{N} = 4$, $D = 4$ Freedman-Schwarz gauged supergravity model [23]. This exact solution is globally regular, preserves 1/4 of the initial supersymmetry of the Freedman-Schwarz model and has unit magnetic charge. Its ten-dimensional lift was shown to represent 5-branes wrapped on a shrinking S^2 [22]. As conjectured by Maldacena and Nuñez, this solution provides a holographic description for $\mathcal{N} = 1$, $D = 4$ super-Yang-Mills theory [24].

The four dimensional Chamseddine-Volkov solution in [21, 22] can be uplifted to $D = 5$ [25] to become a solution of a consistent truncation of the $\mathcal{N} = 4$ Romans' model [26] with an action

$$I_5 = \frac{1}{4\pi} \int d^5x \sqrt{-g} \left(\frac{1}{4} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{2\sqrt{2/3}\phi} F_{\mu\nu}^I F^{I\mu\nu} + \frac{1}{8} e^{-2\sqrt{2/3}\phi} \right), \quad (21)$$

with $F_{\mu\nu}^I$ the SU(2) YM field strength. The uplifted Chamseddine-Volkov solution reads [25]

$$ds^2 = r_0^2 e^{2\nu} \left(-dt^2 + dr^2 + Y(d\theta^2 + \sin^2 \theta d\varphi^2) + (dx^5)^2 \right), \text{ with } Y = 2r \coth r - \frac{r^2}{\sinh^2 r} - 1, \quad e^{6\nu} = \frac{\sinh^2 r}{Y}, \quad (22)$$

r_0 being an integration constant, a dilaton field

$$\phi = \phi_0 + \sqrt{\frac{3}{2}}\nu \quad (23)$$

and a SU(2) field given by (7), with $w = r/\sinh r$. This configuration is neither asymptotically AdS nor asymptotically flat, a common situation in the presence of a Liouville dilaton potential [27, 28].

To generate a nontrivial $D = 4$ Melvin-type solution, one twists again the five dimensional configuration $\varphi \rightarrow \varphi + B_0 x^5$, and considers the KK reduction along the x^5 -direction. Thus we find that (21) leads to the four dimensional action, which different from (1), contains two dilatons with a nontrivial potential

$$I_4 = \frac{1}{4\pi} \int d^4x \sqrt{-\gamma} \left[\frac{\mathcal{R}}{4} - \frac{1}{2} \nabla_i \psi \nabla^i \psi - \frac{1}{2} \nabla_i \phi \nabla^i \phi - e^{2\sqrt{3}\psi} \frac{1}{4} f_{ij} f^{ij} - e^{2\psi/\sqrt{3}+2\sqrt{2/3}\phi} \frac{1}{4} \mathcal{F}'^I{}_{ij} \mathcal{F}'^{Iij} \right. \\ \left. - e^{-4\psi/\sqrt{3}+2\sqrt{2/3}\phi} \frac{1}{4} D_i \Phi^I D^i \Phi^I + \frac{1}{8} e^{-2\sqrt{2/3}\phi-2\psi/\sqrt{3}} \right], \quad (24)$$

(one can see that (24) differs also from the bosonic truncation of the $\mathcal{N} = 4$, $D = 4$ Freedman-Schwarz gauged supergravity model used in [21, 22]).

The four-dimensional line element reads

$$ds^2 = r_0^3 e^{3\nu} \sqrt{\Lambda} \left(-dt^2 + dr^2 + Y(d\theta^2 + \frac{\sin^2 \theta d\varphi^2}{\Lambda}) \right), \quad \text{with } \Lambda = 1 + B_0^2 Y \sin^2 \theta, \quad (25)$$

while the expression of the new dilaton ψ and the nonvanishing U(1) potential is

$$e^{a\psi} = r_0 e^\nu \sqrt{\Lambda}, \quad \mathcal{W}_i dx^i = \frac{B_0 Y \sin^2 \theta}{2\Lambda} d\varphi. \quad (26)$$

The four-dimensional YM and Higgs fields are still given by (16), (17), with $w = r/\sinh r$.

One can easily see that for $B_0 = 0$ the Chamseddine-Volkov solution is recovered, since the scalars ϕ, ψ are not independent in this case. Asymptotically, the geometry (25) approaches the Melvin-type solution in $\mathcal{N} = 4$, $D = 4$ gauged supergravity found in [29]. Therefore we interpret the solution (25)-(26) as describing a nonabelian soliton in a magnetic universe. The same procedure can be applied to the more general globally regular and black hole solutions in [30].

5 Further remarks

The main purpose of this paper was to propose a generalization of the known $D = 4$ spherically symmetric nonabelian solutions by including the effects of a background U(1) magnetic field. In this case, the resulting configurations have axial symmetry and approach asymptotically a dilatonic Melvin background. In our approach, we have used a twisting procedure applied to a set of five-dimensional configurations in EYM theory. It would be interesting to construct this type of solutions for a simpler version of the action than (1), without making use of the twisting procedure; however, this would require to solve a complicated set of partial differential equations with suitable boundary conditions.

More complicated solutions with Melvin-type asymptotics in EYM-Higgs-U(1)-dilaton theory are found by starting with other static EYM configurations instead of (6), (7). The general procedure works as follows: one starts with an axially symmetric EYMd ($a = \sqrt{3}$) solution $(\gamma_{ij}^0, A_i^{(0)I}, \psi^0)$, where $\gamma_{ij}^0 dx^i dx^j =$

$d\ell^2 + \gamma_{\varphi\varphi}^0 d\varphi^2$, and uplifts it to $D = 5$ according to (11). After twisting and reducing back to four dimensions, one generates in this way a new configuration with

$$ds^2 = \gamma_{ij} dx^i dx^j = \sqrt{\Lambda} (d\ell^2 + \frac{\gamma_{\varphi\varphi}^0}{\Lambda} d\varphi^2), \quad \text{with } \Lambda = 1 + e^{-3a\psi_0} B_0^2 \gamma_{\varphi\varphi}^0, \quad e^{2a\psi} = e^{2a\psi_0} \Lambda, \quad (27)$$

$$\mathcal{W}_i dx^i = \frac{e^{-3a\psi} B_0}{2\Lambda} \gamma_{\varphi\varphi}^0 d\varphi, \quad \Phi^I = B_0 A_\varphi^{(0)I}, \quad A_i^I dx^i = A_i^{(0)I} dx^i - 2B_0 A_\varphi^{(0)I} \mathcal{W}_i dx^i.$$

For example, the $D = 4$ EYM(-dilaton) theory possesses also static axially symmetric black hole solutions [19, 31], with (these configurations are not known in closed form)

$$d\ell^2 = -f dt^2 + \frac{m}{f} dr^2 + \frac{mr^2}{f} d\theta^2, \quad \gamma_{\varphi\varphi}^0 = \frac{lr^2 \sin^2 \theta}{f}, \quad (28)$$

where the metric functions f , m and l are functions of the coordinates r and θ , only. After a suitable gauge transformation, the $SU(2)$ matter fields of these solutions are written in terms of four potentials $H_i(r, \theta)$ as

$$A_i^{(0)} dx^i = n \sin \theta (H_3 \tau_3 + (1 - H_4) \tau_1) d\varphi - ((H_1/r) dr + (1 - H_2) d\theta) \tau_2 + \tau_2 d\theta + n \tau_3 \cos \theta d\varphi - n \tau_1 \sin \theta d\varphi. \quad (29)$$

These asymptotically flat solutions are characterized by their horizon radius and three positive integers (k, n, p) , where k is related to the polar angle, n to the azimuthal angle and p to the node number of some gauge functions (the spherically symmetric solutions have $k = n = 1$). As $r_h \rightarrow 0$, a nontrivial globally regular solution is approached [32]. By using this type of seed solutions one can construct more general axially symmetric configurations describing vortex ring solutions in a background $U(1)$ magnetic field, where these vortex ring solutions need not only be located in the xy -plane, but might also come in pairs located symmetrically above and below the xy -plane (similar to the asymptotically flat vortex ring solutions [17]). Similar to the asymptotically flat case, one expects all these configurations to be unstable.

Fluxbrane solutions with nonabelian fields in $4 + N$ spacetime dimensions can be generated in a similar way, by starting again with solutions of the Eqs. (8) (the dilaton coupling constant there would depend on N). Also, a similar construction to that presented in this paper can be done starting with a more complicated higher dimensional action instead of (9), a particularly interesting case being the $D = 10$ low energy heterotic string theory action, which contains nonabelian fields in the bulk.

The Einstein-Maxwell-dilaton theory has also a solution describing a pair of oppositely charged black holes in an external gauge field [33, 34]. Its euclideanised version describes the analogue of the Schwinger pair production of charged particles in a uniform electromagnetic field [34]. It would be interesting to construct the nonabelian counterparts of these configurations.

Acknowledgement

BK gratefully acknowledges support by the German Aerospace Center. The work of ER was supported by a fellowship from the Alexander von Humboldt Foundation.

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